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Finite amplitude side-band stability of a viscous film

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The nonlinear stability of a viscous film flowing steadily down an inclined plane is investigated by the method of multiple scales. It is shown that the supercritically stable, finite amplitude, long, monochromatic wave obtained by Lin (1969, 1970, 1971) is stable to side-band disturbances under modal interaction if the bandwidth is less in magnitude than ϵ, the ratio of the amplitude to the film thickness. Near the upper branch of the linear neutral-stability curve where the amplification rate c_i is $O(\epsilon^2)$, the nonlinear evolution of initially infinitesimal waves of a finite bandwidth is shown to obey the Landau–Stuart equation. Near the lower branch of the neutral curve, the nonlinear evolution is stronger. An equation is derived for describing this strong nonlinear development of relatively long waves. In practice, disturbance of this type clusters in the form of a hump which cannot be constructed only by the first few harmonics.

1. Introduction

The problem of finite amplitude stability of a viscous liquid layer flowing down an inclined plane has been studied by Benney (1966), Lin (1969, 1970) and Gjevik (1970). The wave motions in such a liquid film were also studied, not in the context of stability, by Kapitza & Kapitza (1949), Anshus (1965), Mei (1966) and many others. A list of useful references can be found in the work of Duckler (1972). In all the stability analyses mentioned above, only disturbances of the same mode were considered. On the other hand, it is a known experimental fact that to control the wave motion precisely at a given mode is an extremely difficult task. In laboratories, the presence of side-band disturbances can hardly be avoided. Thus, the question naturally arises whether such a filtered finite amplitude wave is stable with respect to side-band disturbances, which, in general, cause 'resonant' modal interactions. Partial answers to this question have been offered recently, for the case of parallel flows with rigid boundaries, by Stewartson & Stuart (1971), Hocking & Stewartson (1971, 1972), Hocking, Stewartson & Stuart (1972) and DiPrima, Eckhaus & Segel (1971). The purpose of this work is to answer the same question for the case of a parallel flow with a free surface and to elucidate the mechanism of nonlinear stability.

The presence of the free surface introduces additional interesting effects of surface tension and gravity. These effects change the character of the instability dramatically in a parallel flow. While the instability of the parallel flow between two rigid walls takes the form of short shear waves the instability in a liquid film takes the form of long gravity–capillary waves at the relatively small Reynolds number. Another interesting feature of the film instability is that there exists no finite critical wavelength according to the linear theory, in contrast to the case of rigid boundaries. The linear theory predicts the instability to take place in the form of an infinitely long wave. On the other hand, surface waves of finite wavelengths were observed, as the consequence of the instability, by Kapitza & Kapitza (1949) and Binnie (1957). This led to the conjecture that the observed waves are the most amplified waves with the wavelength λ_m predicted by the linear theory. However, referring to the case of a vertical film, Benjamin (1957) pointed out that "one can scarcely expect waves to appear with a strictly uniform and distinct periodicity, because under all conditions infinitesimal waves with a wide range of wavelengths are unstable, and the wave with length λ_m comes into prominence only through a rather uncritical selection process depending on differences in the rates of amplification of different wavelengths. The ultimate state of the amplified waves is, of course, determined largely by nonlinear effects which remain unknown". This statement is consistent with the experiment of Kapitza & Kapitza (1949), who found that distinctively periodic waves could not be observed unless the disturbances were introduced at precisely controlled frequencies. Thus, the present author (1969, 1970) was led to investigate the nonlinear evolution of the Benjamin–Yih wave of a given mode. In the present work, the nonlinear instability to disturbances of a finite frequency bandwidth is studied.

In the next section the equation of motion of the free surface is obtained. On the basis of this equation the stability of a viscous liquid film with respect to disturbances of finite amplitude and finite bandwidth is then investigated by the method of multiple scales. The nonlinear evolution near the upper branch of the neutral curve of the initially infinitesimal but exponentially growing waves of different modes within a narrow band is shown to obey the modified Landau–Stuart equation. Near the lower branch of the neutral curve the modal interaction is stronger and the corresponding nonlinear development cannot be adequately described by the modified Landau–Stuart equation. An equation which describes this strong modal interaction is derived.

2. Equation of the free surface

The following derivation of the governing equation of the free surface follows closely the formulation of Benney (1966). Consider a layer of an incompressible viscous fluid flowing down a plane inclined at an angle β to the horizontal. The motion of the fluid is governed by

$$\nabla . \mathbf{V} = 0, \tag{1}$$

$$\frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V} . \nabla)\, \mathbf{V} = -\frac{1}{\rho} \nabla p + \nu \nabla^2 \mathbf{V} + \mathbf{g}, \tag{2}$$

where $\mathbf{V}$ is the velocity vector, t the time, p the pressure, $\mathbf{g}$ the gravitational acceleration, ν the kinematic viscosity, ρ the density, ∇ the gradient operator and ∇^2 the Laplacian. Solution of the above equations with the conditions of no slip at the plane and zero stress at the free surface gives the following primary flow:

$$\overline{u} = (g \sin\beta/2\nu)(2h_0 y - y^2) \quad (\overline{v} = 0),$$

$$\overline{p} = p_0 - \rho g \cos\beta(y - h_0),$$

where $\overline{u}$ and $\overline{v}$ are respectively the velocity components in the directions parallel to and normal to the bottom plane, $\overline{p}$ the pressure in the film, p_0 the atmospheric pressure, h_0 the constant film thickness and y the distance measured perpendicularly upward from the inclined plane. The stability of the above primary flow is again governed by (1) and (2). The boundary conditions are the no-slip condition at the plane and the vanishing of the tangential stress p_s and the total normal force per unit area at the free surface $y = h(x)$, i.e.

$$p_s = 0,$$

$$p_0 + p_n - Th_{xx}/(1+h_x^2)^{\frac{3}{2}} = 0,$$

where T is the surface tension, p_n the normal stress and the subscript x denotes partial differentiation with respect to the distance in the direction of the flow. Similarly, the subscript y will be used in the following to denote partial differentiation with respect to y. p_s and p_n are related to the velocity field through the Cartesian stress tensors by

$$p_s = p_{12}\cos 2\gamma + \tfrac{1}{2}(p_{22} - p_{11})\sin 2\gamma,$$

$$p_n = p_{22}\cos^2\gamma + p_{11}\sin^2\gamma - p_{12}\sin 2\gamma,$$

$$\tan\gamma = h_x,$$

$$p_{11} = -p + 2\rho\nu u_x, \quad p_{22} = -p + 2\rho\nu v_y, \quad p_{12} = \rho\nu(u_y + v_x),$$

where u and v are the x and y components of the velocity vector $\mathbf{V}$. In addition, the kinematic boundary condition at the free surface

$$h_t + (\overline{u} + u')h_x - v' = 0$$

must also be satisfied. In the above equation, primes denote perturbations from the primary flow $\overline{u}$.

Introducing a perturbation stream function ψ' which satisfies the continuity equation (1), i.e. $u' = \psi'_y$ and $v' = -\psi'_x$, and then normalizing x with $\lambda/2\pi$, λ being the wavelength, y and h with h_0, $\overline{p}$ and its perturbation p' with $\rho g h_0 \sin\beta$, t with $\lambda/2\pi U$, where $U = u_{\max}$, and ψ' with Uh_0, we have from (2) and its boundary conditions the following set of equations:

$$\psi_{yyyy} = \alpha R\{\psi_{yyt} + (\overline{u} + \psi_y)\psi_{xyy} - (\overline{u}_{yy} + \psi_{yyy})\psi_x\} - 2\alpha^2\psi_{xxyy}$$
$$+ \alpha^3 R\{\psi_{xxt} + (\overline{u} + \psi_y)\psi_{xxx} - \psi_x\psi_{xxy}\} - \alpha^4\psi_{xxxx}, \quad (3)$$

$$\psi_x = \psi_y = 0 \quad \text{at} \quad y = 0, \qquad (4), (5)$$

$$h_t + (\overline{u} + \psi_y)h_x + \psi_x = 0 \quad \text{at} \quad y = h, \qquad (6)$$

$$(\bar{u}_y + \psi_{yy} - \alpha^2 \psi_{xx})(1-\alpha^2 h_x^2) - 4\alpha^2 \psi_{xy} h_x = 0 \quad \text{at} \quad y = h, \tag{7}$$

$$-\frac{W\alpha^2 h_{xx}}{(1+\alpha^2 h_x^2)^{\frac{3}{2}}} + (h-1)\cot\beta - p - \alpha\psi_{xy}\frac{1+\alpha^2 h_x^2}{1-\alpha^2 h_x^2} = 0 \quad \text{at} \quad y = h, \tag{8}$$

$$p_x = (2\alpha)^{-1}\psi_{yyy} - \tfrac{1}{2}R\{\psi_{yt} + (\bar{u}+\psi_y)\psi_{xy} - (\bar{u}_y + \psi_{yy})\psi_x\} + \tfrac{1}{2}\alpha\psi_{xxy}, \tag{9}$$

$$\bar{u} = 2y - y^2, \tag{10}$$

where

$$R = \frac{Uh_0}{\nu} = \frac{gh_0^3 \sin\beta}{2\nu^2}, \quad W = \frac{T}{\rho g h_0^2 \sin\beta}, \quad \alpha = \frac{2\pi h_0}{\lambda}$$

are respectively the Reynolds number, the Weber number and the number of waves in a distance $2\pi h_0$. It should be pointed out that all variables appearing in (3)–(10) are dimensionless and should not be confused with the dimensional variables appearing in equations prior to (3). The above set of equations (3)–(10) is identical to that obtained by Benney (1966) except for (8), corresponding to Benney's equation (28), in which there is an error in the first term.

The solution to the above system will be expanded in terms of the small parameter α (wavelength long compared with film thickness) as follows:

$$\psi(x,y,t) = \sum_{n=0} \alpha^n \psi^{(n)}(x,y,t),$$

with

$$\psi^{(n)} = \sum_{m=0} A_m^{(n)}(x,t)\, y^m.$$

The first two coefficients in the above series are chosen to be zero so that the boundary conditions (4) and (5) are satisfied. The rest of the coefficients are determined by demanding that the resulting series satisfies (3), (7), (8) and (9) order by order. Substitution of the series solution thus obtained up to $O(\alpha^2)$ into the kinematic boundary condition (6) then leads to (cf. Benney 1966)

$$h_t + A(h)h_x + \alpha\frac{\partial}{\partial x}[B(h)h_x + C(h)h_{xxx}] + \alpha^2\frac{\partial}{\partial x}[D(h)h_x^2 + E(h)h_{xx} + F(h)h_{xxxx} + G(h)h_x h_{xxx} + H(h)h_{xx}^2 + I(h)h_x^2 h_{xx}] + O(\alpha^3) = 0, \tag{11}$$

where

$$A(h) = 2h^2, \quad B(h) = \tfrac{8}{15}Rh^6 - \tfrac{2}{3}\cot\beta h^3, \quad C(h) = \tfrac{2}{3}\alpha^2 W h^3,$$
$$D(h) = \tfrac{1016}{315}R^2h^9 + \tfrac{14}{3}h^3 - \tfrac{32}{15}R\cot\beta h^6,$$
$$E(h) = \tfrac{32}{63}R^2h^{10} + 2h^4 - \tfrac{40}{63}R\cot\beta h^7,$$
$$F(h) = \tfrac{40}{63}\alpha^2 RWh^7, \quad G(h) = \tfrac{16}{3}\alpha^2 RWh^6,$$
$$H(h) = \tfrac{16}{5}\alpha^2 RWh^6, \quad I(h) = \tfrac{32}{5}\alpha^2 RWh^5.$$

The above results have been checked independently by Mr M. V. Krishna. Equation (11) with $W = 0$ in its coefficients corresponds to Benney's equations (38)–(44). Upon comparison we found that the second terms of the right sides of his equations (43) and (44) contain minor algebraic errors. They should be $-\frac{3632}{315}R^2h^9$ and $-\frac{1016}{35}R^2h^8$ respectively.

3. Modal interactions

While (11) is valid for long waves of arbitrary amplitude, solutions are not at all easy to obtain. Therefore we shall confine ourselves to the case of weakly nonlinear wave motion which perturbs the free surface only slightly. Thus, we write

$$h = 1+\eta, \quad \eta \ll 1.$$

Substituting the above into (11) and expanding the coefficients in (11) about $y = 1$, we obtain

$$\begin{aligned} L_0\eta = & -(A'\eta+\tfrac{1}{2}A''\eta^2)\,\eta_x - \alpha\frac{\partial}{\partial x}[(B'\eta+\tfrac{1}{2}B''\eta^2)\,\eta_x + (C'\eta+\tfrac{1}{2}C''\eta^2)\,\eta_{xxx}] - \alpha^2\frac{\partial}{\partial x} \\ & \times[(D+D'\eta+\tfrac{1}{2}D''\eta^2)\,\eta_x^2 + (E'\eta+\tfrac{1}{2}E''\eta^2)\,\eta_{xx} + (F'\eta+\tfrac{1}{2}F''\eta^2)\,\eta_{xxxx} \\ & + (G+G'\eta+\tfrac{1}{2}G''\eta^2)\,\eta_x\eta_{xxx} + (H+H'\eta+\tfrac{1}{2}H''\eta^2)\,\eta_{xx}^2 \\ & + (I+I'\eta+\tfrac{1}{2}I''\eta^2)\,\eta_x^2\eta_{xx}] + O(\alpha^3\eta, \eta^4), \end{aligned} \tag{12}$$

where $$L_0 = \frac{\partial}{\partial t} + A\frac{\partial}{\partial x} + \alpha\left(B\frac{\partial^2}{\partial x^2} + C\frac{\partial^4}{\partial x^4}\right) + \alpha^2\left(E\frac{\partial^3}{\partial x^3} + F\frac{\partial^5}{\partial x^5}\right)$$

and terms $O(\eta^3)$ and higher are omitted in the Taylor series expansions; primes denote differentiation with respect to h and all coefficients, A, B and their derivatives etc., stand for $A(1)$, $B(1)$, etc. Neglecting the nonlinear terms in (12), we obtain the governing equation for the linear stability problem:

$$L_0\eta = 0.$$

This equation has the normal-mode solution

$$\eta = \Gamma\exp[i(x-ct)] + \bar{\Gamma}\exp[-i(x-\bar{c}t)], \tag{13}$$

where the overbars denote the complex conjugate, Γ is an arbitrary multiplication factor independent of x and t, and c is the eigenvalue, given by

$$c = c_r + ic_i = 2 + i\alpha(B-C) - \alpha^2(E-F).$$

Noting that $\alpha^2 W = O(1)$ in laboratory situations and substituting the values for B, C, E and F into the above equation, we have

$$\begin{aligned} c_r &= 2-\alpha^2(\tfrac{32}{63}R^2 + 2 - \tfrac{40}{63}R\cot\beta - \tfrac{40}{63}\alpha^2 RW), \\ c_i &= \alpha(\tfrac{8}{15}R - \tfrac{2}{3}\cot\beta - \tfrac{2}{3}\alpha^2 W), \end{aligned}$$

which is the eigenvalue obtained by Benjamin (1957) and Yih (1963) from the solution to the Orr–Sommerfeld equation. Note from the expression for the wave speed c_r that the long waves in a liquid film are only weakly dispersive and travel at approximately twice the speed of the unperturbed surface. c_i is the linear amplification factor. The infinitesimal disturbances grow or decay exponentially according as $c_i \gtrless 0$. $c_i = 0$ gives the linear neutral-stability curve, which consists of the straight line $\alpha = 0$ and the parabola

$$\tfrac{8}{15}R - \tfrac{2}{3}\cot\beta - \tfrac{2}{3}\alpha^2 W = 0.$$

These two lines intersect at the bifurcation point $\alpha = 0$, $R = \frac{5}{4}\cot\beta = R_c$. The neutral curve for water at 15 °C with $\beta = 90°$ is shown in figure 1.

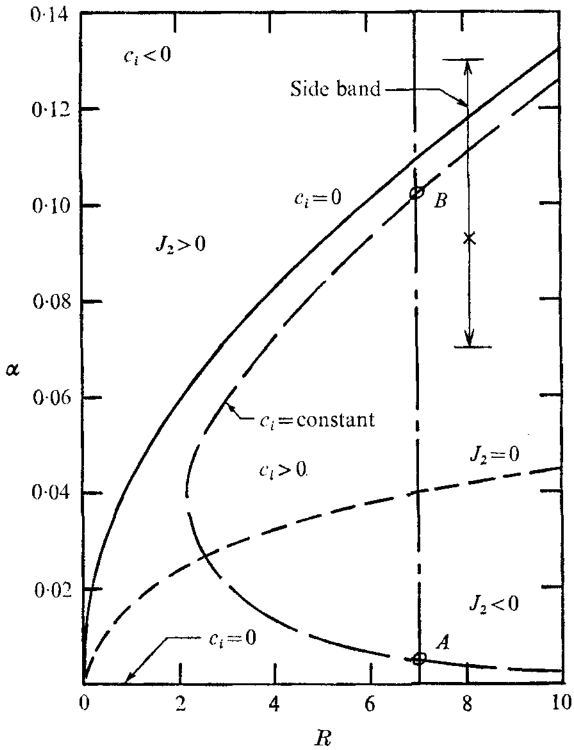

FIGURE 1. Stability curves; $\beta = 90°$, $W = 463{\cdot}3$. ×, Kapitza's experiment.

Benjamin (1961) showed that the asymptotic behaviour at large times of linear two-dimensional dispersive waves in an unstable liquid film is given by

$$(\Gamma\bar{\Gamma})^{\frac{1}{2}} \sim t^{-\frac{1}{2}} \exp\left[-\frac{x'^2}{at}+\frac{\alpha_m}{\alpha}c_i t\right],$$

where a is a positive constant and x' is the distance measured in a frame moving at the linear group velocity. The same asymptotic behaviour was found by Stewartson & Stuart (1971) for plane Poiseuille flow. The above expression states that while the amplitude of the linear wave is amplified by a factor $c_i^{\frac{1}{2}} e$ in a time of order c_i^{-1} the dispersion of the waves leads to the formation of a wave packet whose characteristic length is $O(c_i^{-\frac{1}{2}})$. Near the linear neutral curve this characteristic time, being $O(c_i^{-1})$, is large. Of course, before this time is reached the nonlinear effects become important. However, around the outskirts of the wave packet, where $x' = O(c_i^{-\frac{1}{2}})$, the wave amplitudes remain small. Thus one expects a slow nonlinear modulation over a time $O(\epsilon^{-2})$ and over a distance $O(\epsilon)$, where ϵ is a small parameter independent of α and such that $c_i = O(\epsilon^2)$. This expectation, together with the anticipation that the wave packet may travel at a group velocity of order one, lead one to define the following slow variables:

$$X = \epsilon x, \quad T_1 = \epsilon t, \quad T = \epsilon^2 t.$$

Now, the wave amplitude will be expanded as

$$\eta(\alpha, x, t, X, T_1, T) = \epsilon\eta_1+\epsilon^2\eta_2+\epsilon^3\eta_3+\dots. \tag{14}$$

Substitution of the above expansion with

$$\frac{\partial}{\partial t} \to \frac{\partial}{\partial t} + \epsilon \frac{\partial}{\partial T_1} + \epsilon^2 \frac{\partial^2}{\partial T}, \quad \frac{\partial}{\partial x} \to \frac{\partial}{\partial x} + \epsilon \frac{\partial}{\partial X}$$

into (12) gives

$$(L_0 + \epsilon L_1 + \epsilon^2 L_2 + \ldots)(\epsilon \eta_1 + \epsilon^2 \eta_2 + \epsilon^3 \eta_3 + \ldots) = \text{nonlinear terms}, \tag{15}$$

in which

$$\left.\begin{aligned} L_1 &= \frac{\partial}{\partial T_1} + A\frac{\partial}{\partial x} + \alpha\left(2B\frac{\partial}{\partial x}\frac{\partial}{\partial X} + 4C\frac{\partial^3}{\partial x^3}\frac{\partial}{\partial X}\right) + \alpha^2\left(3E\frac{\partial^2}{\partial x^2}\frac{\partial}{\partial X} + 5F\frac{\partial^4}{\partial x^4}\frac{\partial}{\partial X}\right), \\ L_2 &= \frac{\partial}{\partial T} + \alpha\left(B\frac{\partial^2}{\partial X^2} + 6C\frac{\partial^2}{\partial x^2}\frac{\partial^2}{\partial X^2}\right) + \alpha^2\left(3E\frac{\partial}{\partial x}\frac{\partial^2}{\partial X^2} + 10F\frac{\partial^3}{\partial x^3}\frac{\partial^2}{\partial X^2}\right). \end{aligned}\right\} \tag{16}$$

Thus, the equation for the $O(\epsilon)$ solution is $L_0\eta_1 = 0$, the solution of which has already been given in (13). However c in (13) must be replaced by c_r, since in the vicinity of the neutral curve where $c_i = O(\epsilon^2)$, the function $\exp(c_i t)$ is slowly varying and must be absorbed in $\Gamma = \Gamma(X, T_1, T)$. We now investigate the nonlinear evolution of the unstable linear waves in the region where $c_i = O(\epsilon^2)$.

It follows from (12)–(15) that the $O(\epsilon^2)$ solution is to be obtained from

$$\begin{aligned} L_0\eta_2 = -\left[\frac{\partial}{\partial T_1} + H_2\frac{\partial}{\partial X}\right]\Gamma\exp[i(x - c_r t)] \\ + Q_1\Gamma^2\exp[2i(x - c_r t)] + \text{complex conjugate}, \end{aligned} \tag{17}$$

where

$$H_2 = H_{2r} + iH_{2i} = [2 + \alpha^2(5F - 3E)] + i2\alpha(B - 2C),$$

$$Q_1 = -4i + 4\alpha[\tfrac{8}{5}R - \cot\beta - \alpha^2 W] + 2i\alpha^2[\tfrac{38}{3} + \tfrac{2616}{315}R^2 - \tfrac{296}{45}R\cot\beta - \tfrac{584}{45}\alpha^2 RW].$$

The first term on the right side of (17) is of the same form as the complementary solution of the equation. This secular term leads to a steady growth of the amplitude over a time $O(\epsilon^{-1})$. The necessary condition for the existence of a finite amplitude periodic motion with slow modulation over $t = O(\epsilon^{-2})$ is then

$$(\partial/\partial T_1 + H_2\partial/\partial x)\Gamma = 0.$$

This equation possesses the normal-mode solution $\Gamma = \zeta(T)\exp[i(X - H_2 T_1)]$. We consider modal interaction near the linear neutral curve such that

$$H_{2i} = 2(c_i - \tfrac{4}{3}\alpha^3 W) = O(\epsilon).$$

For all experiments cited at the end of this section, $H_{2i} = O(\epsilon)$. This being the case, $\exp[H_{2i}T_1]$ varies as slowly as $\exp[c_i t]$ and can be absorbed in $\zeta(T)$. Then the secular condition leads to the requirement that the wave envelope must propagate at the speed H_{2r}, which turns out to be the group velocity, since

$$d(\alpha c_r)/d\alpha = 2 + (5F - E)\alpha^2 = H_{2r}.$$

That is to say Γ must have the functional form

$$\Gamma = \Gamma(X - H_{2r}T_1, T).$$

With the secular condition satisfied, the solution of (17) is easily found to be

$$\eta_2 = H_1\Gamma^2 \exp[2i(x-c_r t)] + \bar{H}_1\bar{\Gamma}^2 \exp[-2i(x-c_r t)],$$

where

$$H_1 = \frac{[2(4C-B)Q_{1r}+3\alpha(5F-E)Q_{1i}]+i[2(4C-B)Q_{1i}-3\alpha(5F-E)Q_{1r}]}{2\alpha[(4C-B)^2+9\alpha^2(5F-E)^2]},$$

in which the subscript r or i stands for the real or imaginary part respectively.

Similarly, from the order ϵ^3 terms, we have

$$\begin{aligned}L_0\eta_3 = &-\epsilon^{-2}L_0\eta_1 - L_1\eta_2 - L_2\eta_1 - A'(\eta_1\eta_{2x}+\eta_2\eta_{1x}) - \tfrac{1}{2}A''\eta_1^2\eta_{1x} - \alpha[B'(\eta_1\eta_{2xx}\\ &+\eta_2\eta_{1xx}) + \tfrac{1}{2}B''\eta_1^2\eta_{1xx} + 2B'\eta_{1x}\eta_{2x} + B''\eta_1\eta_{1x}\eta_{1x} + C'(\eta_1\eta_{2xxxx}+\eta_2\eta_{1xxxx})\\ &+\tfrac{1}{2}C''\eta_1^2\eta_{1xxxx} + C'(\eta_{1x}\eta_{2xxx}+\eta_{2x}\eta_{1xxx}) + C''\eta_1\eta_{1x}\eta_{1xxx}].\end{aligned}$$

It should be pointed out that there are $O(\epsilon^3)$ contributions from $L_0\epsilon\eta_1$. The term $\alpha(B\partial^2/\partial x^2 + C\partial^4/\partial x^4)$ in L_0 leads to

$$-\epsilon\alpha(B-C)[\Gamma\exp\{i(x-c_r t)\} + \bar{\Gamma}\exp\{-i(x-c_r t)\}],$$

which is $O(\epsilon^3)$ in the vicinity of the linear neutral curve where $c_i = O(\epsilon^2)$, since $\alpha(\beta - C) = c_i$. Some of the modal interactions indicated on the right side of the above equation lead to secular terms. Imposing the secular condition, we have

$$\begin{aligned}-c_i\epsilon^{-2}\Gamma + L_2\eta_1 + L_1\eta_1 + i(A'H_1 + \tfrac{1}{2}A'')\Gamma^2\bar{\Gamma} + \alpha(-B'H - \tfrac{1}{2}B''&\\ +10C'H_1 + \tfrac{1}{2}C'')\Gamma^2\bar{\Gamma} &= 0.\end{aligned}$$

Applying the operators L_1 and L_2, we can write the above equation as

$$\partial\Gamma/\partial T + J_1\partial^2\Gamma/\partial X^2 - c_i'\Gamma + (J_2 + iJ_4)\Gamma^2\bar{\Gamma} = 0, \tag{18}$$

where

$$c_i' = \epsilon^{-2}c_i$$

and

$$\begin{aligned}\Gamma &= \Gamma[(X - H_{2r}T_1), T],\\ J_1 &= \alpha(B-6C),\\ J_2 &= -A'H_{1i} + \alpha[\tfrac{1}{2}(C''-B'') + (10C'-B')H_{1r}],\\ J_4 &= A'H_{1r} + \tfrac{1}{2}A'' + \alpha H_{1i}(10C'-B').\end{aligned}$$

Equation (18) has been derived by DiPrima *et al.* (1971), Stewartson & Stuart (1971), Newell & Whitehead (1969) and Segel (1969) for other flows. This equation with $c_i = 0$ also describes a light beam in a nonlinear medium (Talanov 1965; Kelly 1965). A complete study of (18) is beyond the scope of the present work. We shall, however, use (18) to study the nonlinear evolution of a filtered wave and its stability to side-band disturbances under modal interactions. For a filtered wave, there is no spatial modulation and the second term in (18) vanishes. The solution of this equation can be written as

$$\Gamma_1 = (\Gamma_1\bar{\Gamma}_1)^{\frac{1}{2}}\exp[iB(T)T]. \tag{19}$$

The initial condition is $\Gamma_1\bar{\Gamma}_1(T) = 0$ at $T = -\infty$. The explicit expressions for $\Gamma_1\bar{\Gamma}_1(T)$ and $B(T)$ are easily found to be

$$\epsilon^2\Gamma_1\bar{\Gamma}_1(T) = \frac{c_i\exp[2c_i'(T-T_0)]}{1 + J_2\exp[2c_i'(T-T_0)]} \tag{20a}$$

and

$$B(T) = -\frac{J_4}{T}\int_{T_0}^{T}\Gamma_1\bar{\Gamma}_1(T)\,dT, \tag{20b}$$

Data	Authors	c_r	r
Water at 15 °C,	Gjevik (1970)	2·21	0·350
$c_i = 0{\cdot}155$,	Kapitza & Kapitza (1949)	1·76	0·160
$\alpha = 0{\cdot}092$,	Present theory	1·81	0·174
$R = 8{\cdot}07$,	Lin (1971)	1·78	0·225
$W = 463{\cdot}3$			
Alcohol at 15 °C,	Kapitza & Kapitza (1949)	1·67	0·163
$c_i = 0{\cdot}174$,	Present theory	1·80	0·178
$\alpha = 0{\cdot}144$,	Lin (1971)	1·73	0·221
$R = 5{\cdot}04$,			
$W = 107{\cdot}2$			
Water at 19 °C,	Binnie (1957)	2·34	Not observed
$c_i = 0{\cdot}121$,	Present theory	1·72	$0{\cdot}149 = [\Gamma_1\bar{\Gamma}_1]^{\frac{1}{2}}$
$\alpha = 0{\cdot}066$,	Lin (1969)	2·01	$0{\cdot}145 = [\Gamma_1\bar{\Gamma}_1]^{\frac{1}{2}}$
$R = 6{\cdot}60$,			
$W = 616{\cdot}7$			

TABLE 1. Waves in liquid films, $\beta = 90°$.

where T_0 is an arbitrary time constant reflecting the arbitrary initial phase. Equations (20) describe the nonlinear modification of the amplitude and the phase speed of the exponentially growing initial disturbances. As $T \to \infty$, an equilibrium amplitude of value c_i/J_2 and the corresponding reduction in wave speed of magnitude $-J_4\Gamma_1\bar{\Gamma}_1(\infty)$ are reached. It should be pointed out that this equilibrium value may be reached only in a region where the values of α, c_i and R are such that J_2 is positive. For example, when the surface tension is zero, $J_2 < 0$ and thus no supercritical stability is possible. On the basis of (20), theoretical predictions are made for comparisons with the experiments of Kapitza & Kapitza and Binnie which were quoted in the work of Lin (1969, 1971).

In the previous work of Lin, the solution was expanded in terms of eigenfunctions. The unmodulated Landau–Stuart equation was assumed to be valid *a priori* and the Landau second coefficient was obtained as an eigenvalue with the aid of a Poincaré eigenvalue stretching of the wave speed. In the present work, the shallow-water expansion is applied to the viscous film following the work of Benney (1966). This expansion formalism allows us to study the modal interaction more readily and yields the nonlinear Schrödinger equation (18) as a necessary condition for supercritical stability.

It may be seen from table 1 that the present theoretical predictions compare very well with both the known theoretical results obtained with an entirely different method and the above-mentioned experiments. It should be pointed out that the values of c_r, c_i and R obtained by Lin (1969, 1971) differ by factors of $\frac{2}{3}$ and $\frac{3}{2}$ respectively from those quoted in table 1. This difference arises from the different normalization factors used in the present analysis. In table 1

$$r = [(1+\eta_{\max})-(1-\eta_{\min})]/[(1+\eta_{\max})+(1-\eta_{\min})],$$

where $\eta_{\max}$ and $\eta_{\min}$ are obtained from the first two terms of (14) with $\Gamma_1\bar{\Gamma}_1 = c_i'/J_2$. The wavenumbers α_m corresponding to the maximum linear amplification rate in each experiment are also given. Note that the waves observed by Kapitza

& Kapitza are not the most amplified waves according to the linear theory. Gjevik's theoretical results are also included for comparison. In Benney's (1966) analysis, the surface tension is neglected and thus no supercritically stable wave motions are predicted. We now proceed to study the stability of these predicted supercritically stable *filtered* waves to side-band disturbances of bandwidth $K\epsilon$. Thus, we let†

$$\Gamma = \Gamma_\infty(T) + [\delta\Gamma_2(T)\exp(iKX) + \delta\Gamma_3(T)\exp(-iKX)]\exp(-iQT),$$

where $Q = c_i' J_4/J_2$, $\Gamma_\infty(T)$ is the limiting solution of (18) as $t \to \infty$ for the filtered wave, i.e. $\Gamma_\infty(T) = (c_i'/J_2)\exp[-iQT]$, and $\delta \ll 1$. Substituting this expression into (18), we obtain from the coefficients of

$$\delta\exp[i(KX - QT)] \quad \text{and} \quad \delta\exp[-i(KX+QT)]$$

$$\frac{\partial}{\partial T}\begin{bmatrix}\Gamma_2\\ \bar{\Gamma}_3\end{bmatrix} + \begin{bmatrix}A_{11} & A_{12}\\ A_{21} & A_{22}\end{bmatrix}\begin{bmatrix}\Gamma_2\\ \bar{\Gamma}_3\end{bmatrix} = 0, \tag{21}$$

where

$$\left.\begin{aligned} A_{11} &= c_i' - J_1K^2 + ic_i'(J_4/J_2), \quad A_{22} = c_i' - J_1K^2 - ic_i'(J_4/J_2),\\ A_{12} &= (J_2 + iJ_4)(c_i'/J_2), \quad A_{21} = (J_2 - iJ_4)(c_i'/J_2).\end{aligned}\right\} \tag{22}$$

Here we consider only the linear stability of the stable filtered wave and write the solution of (21) as

$$\begin{bmatrix}\Gamma_2\\ \bar{\Gamma}_3\end{bmatrix} = \begin{bmatrix}c_2\\ c_3\end{bmatrix} e^{\lambda_1 T},$$

where c_2 and c_3 are constants which are indeterminate within the framework of linear analysis. A nonlinear analysis of Γ_2 and Γ_3 is not intended here. The eigenvalue λ_1 in the above solution is given by

$$\lambda_1 = \tfrac{1}{2}[-(A_{22}+A_{11}) \pm \{(A_{22}+A_{11})^2 - 4(A_{11}A_{22} - A_{21}A_{12})\}^{\frac{1}{2}}].$$

The condition for stability to the side-band disturbance is $\lambda_1 < 0$. It follows from (22) that this condition can be written as

$$J_1K^2 < c_i' \mp c_i'. \tag{23}$$

From the definition of J_1, we have

$$J_1 = c_i - 5\alpha C = c_i - \tfrac{10}{3}\alpha^3 W.$$

For all three experiments listed in table 1, $J_1 < 0$. Thus the above inequality is satisfied for all $K = O(1)$. Consequently, the observed waves are stable to side-band disturbances of bandwidth $O(\epsilon)$. Hocking & Stewartson (1972) found numerically in their study of plane Poiseuille flow that the modal solution of (18) appears to be stable when $J_1 < 0$ and $J_4/J_2 \sim 1$. However, they also found that the stable modal solution with $J_1 < 0$ is never approached if $J_4/J_2 \sim 10$. This is inconsistent with our result (23), which states that if $J_1 < 0$ the stable modal solution can be reached regardless of the value of J_4/J_2. This discrepancy probably arises from the fact that the initial disturbances used in their numerical computation are of finite amplitude while the side-band disturbances used in this study

† The exponential factor in this expression was suggested by Professor K. Stewartson to remove an error in the original manuscript.

are infinitesimal. This difficulty does not arise if we apply (23) only to the liquid film where $J_4/J_2 \sim 1$ according to the explicit expressions of J_4 and J_2 in (18). Then both (23) and the numerical study of Hocking & Stewartson predict modal stability for $J_1 < 0$ and $J_4/J_2 \sim 1$.

For sufficiently long waves, J_1 may become positive. However, it is shown in the next section that, when $J_1 > 0$, (18) is no longer appropriate for describing the nonlinear evolution of such long waves.

4. Long waves

It is seen from the expression J_2 that near $\alpha = 0$

$$\alpha J_2 \sim -\tfrac{128}{3}\left[\tfrac{1}{5}(R-R_c)-\alpha^2 W\right]^{-1}. \tag{24}$$

Thus, at $\alpha = 0$, J_2 is negative as long as $R-R_c > \delta$, $\delta \to 0+$. This implies that the Landau limit does not exist at $\alpha = 0$. Therefore, it is necessary to look more closely into the region where $\alpha \to 0$. By use of simple algebra, it can be shown that for a given flow the curves $R =$ constant and $c_i =$ a sufficiently small constant intersect, in general at two points A and B (cf. figure 1). The point A lies outside and the point B lies inside the region where the Landau limit exists, i.e. $J_2 < 0$ at A and $J_2 > 0$ at B. It can also be shown that $J_1 < 0$ at B and $J_1 > 0$ at A. Now, since B is sufficiently close to the upper branch of the neutral curve such that

$$c_i = O(\epsilon^2), \quad J_1 < 0 \quad \text{and} \quad J_2 > 0,$$

then as was shown in the last section not only is the filtered wave at the point B supercritically stable but also this filtered wave is stable to side-band disturbances with bandwidth $O(\epsilon)$. On the other hand at point A, even if A is so close to the lower branch of the neutral curve $\alpha = 0$ that $c_i = O(\epsilon^2)$, there can be no supercritically stable solution to (18) since $J_2 < 0$ and $J_1 > 0$. Thus we are led to look for equations which describe the nonlinear evolution of these long waves. Note that the expansion for η is singular at $\alpha = 0$ if we regard ϵ and α as independent parameters, since $\eta_2 \sim 1/\alpha$. However if ϵ were dependent on α, $\epsilon = O(\alpha^n)$ say, then the expansion obtained for η would be bounded as $\alpha \to 0$ if $n > 1$. Thus in the region $\alpha \to 0$, the order of magnitude of ϵ relative to α must be determined *a priori*. Consider the case $\epsilon = \alpha^2$. Then to order ϵ, we have from (12)

$$(\partial/\partial t + 2\partial/\partial x)\,\eta_1 = 0.$$

The solution of the above equation can be written as

$$\eta_1 = f(\theta, T),$$

where $\theta = x-2t$ and $T = \alpha^n t$ with $n > 1$. The second-order solution must satisfy

$$\alpha^{n-2}\frac{\partial f}{\partial T} + \frac{8}{15}(R-R_1)\,\alpha^{-1}\frac{\partial^2 f}{\partial x^2} + \tfrac{2}{3}\alpha W\frac{\partial^4 f}{\partial x^4} + E\frac{\partial^3 f}{\partial x^3} + F\frac{\partial^5 f}{\partial x^5} + 4f\frac{\partial f}{\partial x} = 0. \tag{25}$$

If $R-R_1 < O(\alpha)$ and thus $c_i = \frac{8}{15}\alpha[R-R_1] < O(\alpha^2) = O(\epsilon)$, the diffusion term in the above equation can be neglected and near the lower branch of the neutral curve the resulting equation with $n = 2$ describes the nonlinear evolution over a time $O(\epsilon^{-1}) = O(\alpha^{-2})$. If $R - R_1 = O(\alpha)$ and thus $c_i = O(\alpha^2) = O(\epsilon)$, the diffusion term must be retained in (25) and the resulting equation with $n = 2$ describes the strong modal interaction of long waves near the neutral curve, where $c_i = O(\epsilon)$. In the limit $\alpha \to 0$ the above equation is reduced to

$$\frac{\partial f}{\partial T} + \tfrac{8}{15}(R - R_1)\,\alpha^{-1}\frac{\partial^2 f}{\partial x^2} + E\frac{\partial^3 f}{\partial x^3} + 4f\frac{\partial f}{\partial x} = 0. \tag{26}$$

Note that no vestige of surface tension can be found in the above equation. This is due to the fact that $\alpha \to 0$ implies $\lambda \to \infty$, which in turn means that the wave elevation changes appreciably only over an infinitely large distance. Consequently the surface curvature approaches zero and the effect of surface tension cannot be exhibited. However there are infinitely long waves (solitary waves for example) whose amplitude varies appreciably over a distance λ_2 much smaller than the wavelength. For these cases λ_2 instead of λ must be used as a normalization factor and as a result $W\alpha$ or even $W\alpha^2$ may be of order one. Therefore, for these waves, (25) with $n = 2$ is a more appropriate equation to use.

Thus (18) is the appropriate equation for the description of the weak nonlinear evolution of relatively short waves near the upper branch of the neutral curve where $c_i = O(\epsilon^2)$. In practice, these waves can be constructed from the first few harmonics as is the case in the experiments cited in the last section. Near the lower branch of the neutral curve $\alpha = 0$, even where $c_i < O(\alpha^2) = O(\epsilon)$ the modal interaction is stronger. Equation (25) or (26) without the diffusion term is then the governing equation of the nonlinear evolution. Farther away from the neutral curve where $c_i = O(\epsilon)$ the diffusion term in (25) or (26) becomes important. For example, the 'single wave' observed and so termed by Kapitza & Kapitza takes the form of a smooth hump. It is obvious that this hump, being far from sinusoidal, takes more than several harmonics to construct. Moreover, because of its extremely long wavelength, the hump must include long-wave components such that $\alpha \to 0$. Thus (25) or (26) is the appropriate equation for the description of the single wave. On the other hand, the waves given in table 1 are of relatively short wavelength such that $\alpha \to [4(R_1 - R_c)/5W]^{\frac{1}{2}}$, the α on the upper branch of the neutral curve. Therefore the modified Landau–Stuart equation is sufficient for the description of these waves as is evident from the good comparison given in table 1 between the theoretical results based on the Landau–Stuart equation and the experimental results. No numerical results based on (25) or (26) are yet available for comparison with the single waves observed by Kapitza & Kapitza. To summarize the analysis given in the last two sections, we use for illustration the numerical results corresponding to the experiment with water of Kapitza & Kapitza. This particular numerical result is displayed in figure 1. The linear neutral curve $c_i = 0$ calculated from the eigenvalue in (13) divides the α, R plane into two regions. According to the linear theory, the film is stable or unstable at any given R depending on whether the wavelength of the disturbance lies above or below the neutral curve. Similarly the curve $J_2 = 0$ divides the α, R plane into two regions. $J_2 > 0$ in the region above the curve $J_2 = 0$. In the rest of the α, R plane $J_2 < 0$. Since the wave observed by Kapitza & Kapitza corresponds to a point lying below $c_i = 0$ but above $J_2 = 0$, where $J_2 > 0$, the filtered wave attains nonlinear stability according to (20). The nonlinear modulation of this filtered wave due to the neighbouring side-band disturbances is governed by (18). Moreover, the filtered wave is stable to side-band disturbances of bandwidth $O(\epsilon)$ according to (23).

Finally we point out that (26) has also been used to describe the wave propagation on liquid-filled elastic tubes (Johnson 1970), the weak shock profile in plasma (Grad & Hu 1967) and the undular bore in an open channel flow (Johnson 1972).

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